

- 60. Wilson Lot Size Formula** One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

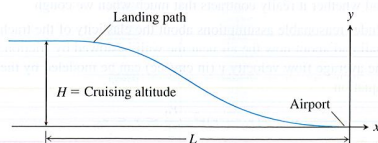
where q is the quantity you order when things run low (shoes, radios, brooms, or whatever the item might be), k is the cost of placing an order (the same, no matter how often you order), c is the cost of one item (a constant), m is the number of items sold each week (a constant), and h is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security).

- (a) Your job, as the inventory manager for your store, is to find the quantity that will minimize $A(q)$. What is it? (The formula you get for the answer is called the *Wilson lot size formula*.)
- (b) Shipping costs sometimes depend on order size. When they do, it is more realistic to replace k by $k + bq$, the sum of k and a constant multiple of q . What is the most economical quantity to order now?
- 61. Production Level** Show that if $r(x) = 6x$ and $c(x) = x^3 - 6x^2 + 15x$ are your revenue and cost functions, then the best you can do is break even (have revenue equal cost).
- 62. Production Level** Suppose $c(x) = x^3 - 20x^2 + 20,000x$ is the cost of manufacturing x items. Find a production level that will minimize the average cost of making x items.

Extending the Ideas

- 63. Airplane Landing Path** An airplane is flying at altitude H when it begins its descent to an airport runway that is at horizontal ground distance L from the airplane, as shown in the figure. Assume that the landing path of the airplane is the graph of a cubic polynomial function $y = ax^3 + bx^2 + cx + d$ where $y(-L) = H$ and $y(0) = 0$.
- (a) What is dy/dx at $x = 0$?
- (b) What is dy/dx at $x = -L$?
- (c) Use the values for dy/dx at $x = 0$ and $x = -L$ together with $y(0) = 0$ and $y(-L) = H$ to show that

$$y(x) = H \left[2 \left(\frac{x}{L} \right)^3 + 3 \left(\frac{x}{L} \right)^2 \right].$$



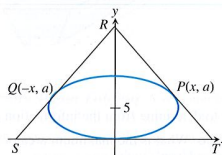
In Exercises 64 and 65, you might find it helpful to use a CAS.

- 64. Generalized Cone Problem** A cone of height h and radius r is constructed from a flat, circular disk of radius a in. as described in Exploration 1.
- (a) Find a formula for the volume V of the cone in terms of x and a .
- (b) Find r and h in the cone of maximum volume for $a = 4, 5, 6, 8$.
- (c) **Writing to Learn** Find a simple relationship between r and h that is independent of a for the cone of maximum volume. Explain how you arrived at your relationship.

- 65. Circumscribing an Ellipse** Let $P(x, a)$ and $Q(-x, a)$ be two points on the upper half of the ellipse

$$\frac{x^2}{100} + \frac{(y-5)^2}{25} = 1$$

centered at $(0, 5)$. A triangle RST is formed by using the tangent lines to the ellipse at Q and P as shown in the figure.



- (a) Show that the area of the triangle is

$$A(x) = -f'(x) \left[x - \frac{f(x)}{f'(x)} \right]^2,$$

where $y = f(x)$ is the function representing the upper half of the ellipse.

- (b) What is the domain of A ? Draw the graph of A . How are the asymptotes of the graph related to the problem situation?
- (c) Determine the height of the triangle with minimum area. How is it related to the y -coordinate of the center of the ellipse?
- (d) Repeat parts (a)–(c) for the ellipse

$$\frac{x^2}{C^2} + \frac{(y-B)^2}{B^2} = 1$$

centered at $(0, B)$. Show that the triangle has minimum area when its height is $3B$.

What you will learn about . . .

- Linear Approximation
- Differentials
- Estimating Change with Differentials
- Absolute, Relative, and Percentage Change
- Sensitivity to Change
- Newton's Method

and why . . .

Engineering and science depend on approximations in most practical applications; it is important to understand how approximation techniques work.

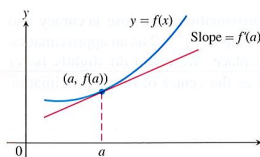


Figure 5.44 The tangent to the curve $y = f(x)$ at $x = a$ is the line $y = f(a) + f'(a)(x - a)$.

5.5 Linearization and Differentials

Linear Approximation

In our study of the derivative we have frequently referred to the “tangent line to the curve” at a point. What makes that tangent line so important mathematically is that it provides a *useful representation of the curve itself* if we stay close enough to the point of tangency. We say that differentiable curves are always **locally linear**, a fact that can best be appreciated graphically by zooming in at a point on the curve, as Exploration 1 shows.

EXPLORATION 1 Appreciating Local Linearity

The function $f(x) = (x^2 + 0.0001)^{1/4} + 0.9$ is differentiable at $x = 0$ and hence “locally linear” there. Let us explore the significance of this fact with the help of a graphing calculator.

- Graph $y = f(x)$ in the “ZoomDecimal” window. What appears to be the behavior of the function at the point $(0, 1)$?
- Show algebraically that f is differentiable at $x = 0$. What is the equation of the tangent line at $(0, 1)$?
- Now zoom in repeatedly, keeping the cursor at $(0, 1)$. What is the long-range outcome of repeated zooming?
- The graph of $y = f(x)$ eventually looks like the graph of a line. What line is it?

We hope that this exploration gives you a new appreciation for the tangent line. As you zoom in on a differentiable function, its graph at that point actually seems to *become* the graph of the tangent line! This observation—that even the most complicated differentiable curve behaves locally like the simplest graph of all, a straight line—is the basis for most of the applications of differential calculus. It is what allows us, for example, to refer to the derivative as the “slope of the curve” or as “the velocity at time t_0 .”

Algebraically, the principle of local linearity means that the *equation* of the tangent line defines a function that can be used to *approximate* a differentiable function near the point of tangency. In recognition of this fact, we give the equation of the tangent line a new name: the *linearization of f at a* . Recall that the tangent line at $(a, f(a))$ has point-slope equation $y - f(a) = f'(x)(x - a)$ (Figure 5.44).

DEFINITION Linearization

If f is differentiable at $x = a$, then the equation of the tangent line,

$$L(x) = f(a) + f'(a)(x - a),$$

defines the **linearization of f at a** . The approximation $f(x) \approx L(x)$ is the **standard linear approximation of f at a** . The point $x = a$ is the **center** of the approximation.

EXAMPLE 1 Finding a Linearization

Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 0$, and use it to approximate $\sqrt{1.02}$ without a calculator. Then use a calculator to determine the accuracy of the approximation.

continued

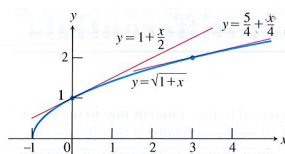


Figure 5.45 The graph of $f(x) = \sqrt{1+x}$ and its linearization at $x = 0$ and $x = 3$. (Example 1)

Why not just use a calculator?

We readily admit that linearization will never replace a calculator when it comes to finding square roots. Indeed, historically it was the other way around. Understanding linearization, however, brings you one step closer to understanding how the calculator finds those square roots so easily. You will get many steps closer when you study Taylor polynomials in Chapter 10. (A linearization is just a Taylor polynomial of degree 1.)

SOLUTION

Since $f(0) = 1$, the point of tangency is $(0, 1)$. Since $f'(x) = \frac{1}{2}(1+x)^{-1/2}$, the slope of the tangent line is $f'(0) = \frac{1}{2}$. Thus

$$L(x) = 1 + \frac{1}{2}(x - 0) = 1 + \frac{x}{2}. \quad (\text{Figure 5.45})$$

To approximate $\sqrt{1.02}$, we use $x = 0.02$:

$$\sqrt{1.02} = f(0.02) \approx L(0.02) = 1 + \frac{0.02}{2} = 1.01$$

The calculator gives $\sqrt{1.02} = 1.009950494$, so the approximation error is $|1.009950494 - 1.01| \approx 4.05 \times 10^{-5}$. We report that the error is less than 10^{-4} .

Now Try Exercise 1.

Look at how accurate the approximation $\sqrt{1+x} \approx 1 + \frac{x}{2}$ is for values of x near 0.

Approximation	[True Value - Approximation]
$\sqrt{1.002} \approx 1 + \frac{0.002}{2} = 1.001$	$< 10^{-6}$
$\sqrt{1.02} \approx 1 + \frac{0.02}{2} = 1.01$	$< 10^{-4}$
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.1$	$< 10^{-2}$

As we move away from zero (the center of the approximation), we lose accuracy and the approximation becomes less useful. For example, using $L(2) = 2$ as an approximation for $f(2) = \sqrt{3}$ is not even accurate to one decimal place. We could do slightly better using $L(2)$ to approximate $f(2)$ if we were to use 3 as the center of our approximation (Figure 5.45).

EXAMPLE 2 Finding a Linearization

Find the linearization of $f(x) = \cos x$ at $x = \pi/2$ and use it to approximate $\cos 1.75$ without a calculator. Then use a calculator to determine the accuracy of the approximation.

SOLUTION

Since $f(\pi/2) = \cos(\pi/2) = 0$, the point of tangency is $(\pi/2, 0)$. The slope of the tangent line is $f'(\pi/2) = -\sin(\pi/2) = -1$. Thus

$$L(x) = 0 + (-1)\left(x - \frac{\pi}{2}\right) = -x + \frac{\pi}{2}. \quad (\text{Figure 5.46})$$

To approximate $\cos(1.75)$, we use $x = 1.75$:

$$\cos 1.75 = f(1.75) \approx L(1.75) = -1.75 + \frac{\pi}{2}$$

The calculator gives $\cos 1.75 = -0.1782460556$, so the approximation error is $|-0.1782460556 - (-1.75 + \pi/2)| \approx 9.57 \times 10^{-4}$. We report that the error is less than 10^{-3} .

Now Try Exercise 5.

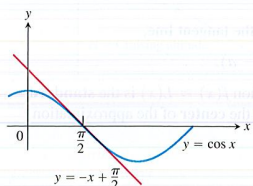


Figure 5.46 The graph of $f(x) = \cos x$ and its linearization at $x = \pi/2$. Near $x = \pi/2$, $\cos x \approx -x + (\pi/2)$. (Example 2)

EXAMPLE 3 Approximating Binomial Powers

Example 1 introduces a special case of a general linearization formula that applies to powers of $1+x$ for small values of x :

$$(1+x)^k \approx 1 + kx$$

If k is a positive integer this follows from the Binomial Theorem, but the formula actually holds for *all* real values of k . (We leave the justification to you as Exercise 7.) Use this formula to find polynomials that will approximate the following functions for values of x close to zero:

$$(a) \sqrt[3]{1-x} \quad (b) \frac{1}{1-x} \quad (c) \sqrt{1+5x^4} \quad (d) \frac{1}{\sqrt{1-x^2}}$$

SOLUTION

We change each expression to the form $(1+y)^k$, where k is a real number and y is a function of x that is close to 0 when x is close to zero. The approximation is then given by $1 + ky$.

$$\begin{aligned} (a) \sqrt[3]{1-x} &= (1+(-x))^{1/3} \approx 1 + \frac{1}{3}(-x) = 1 - \frac{x}{3} \\ (b) \frac{1}{1-x} &= (1+(-x))^{-1} \approx 1 + (-1)(-x) = 1 + x \\ (c) \sqrt{1+5x^4} &= ((1+5x^4))^{1/2} \approx 1 + \frac{1}{2}(5x^4) = 1 + \frac{5}{2}x^4 \\ (d) \frac{1}{\sqrt{1-x^2}} &= ((1+(-x^2))^{-1/2}) \approx 1 + \left(-\frac{1}{2}\right)(-x^2) = 1 + \frac{1}{2}x^2 \end{aligned}$$

Now Try Exercise 9.

EXAMPLE 4 Approximating Roots

Use linearizations to approximate (a) $\sqrt{123}$ and (b) $\sqrt[3]{123}$.

SOLUTION

Part of the analysis is to decide where to center the approximations.

(a) Let $f(x) = \sqrt{x}$. The closest perfect square to 123 is 121, so we center the linearization at $x = 121$. The tangent line at $(121, 11)$ has slope

$$f'(121) = \frac{1}{2}(121)^{-1/2} = \frac{1}{2} \cdot \frac{1}{\sqrt{121}} = \frac{1}{22}.$$

So

$$\sqrt{123} \approx L(123) = 11 + \frac{1}{22}(123 - 121) = 11.09.$$

(b) Let $f(x) = \sqrt[3]{x}$. The closest perfect cube to 123 is 125, so we center the linearization at $x = 125$. The tangent line at $(125, 5)$ has slope

$$f'(125) = \frac{1}{3}(125)^{-2/3} = \frac{1}{3} \cdot \frac{1}{(\sqrt[3]{125})^2} = \frac{1}{75}.$$

So

$$\sqrt[3]{123} \approx L(123) = 5 + \frac{1}{75}(123 - 125) = 4.973.$$

A calculator shows both approximations to be within 10^{-3} of the actual values.

Now Try Exercise 11.

Differentials

Leibniz used the notation dy/dx to represent the derivative of y with respect to x . The notation *looks* like a quotient of real numbers, but it is really a *limit* of quotients in which both

Leibniz and His Notation

Although Leibniz did most of his calculus using dy and dx as separable entities, he never quite settled the issue of what they were. To him, they were “infinitesimals”—nonzero numbers, but infinitesimally small. There was much debate about whether such things could exist in mathematics, but luckily for the early development of calculus it did not matter. Thanks to the Chain Rule, dy/dx behaved like a quotient whether it was one or not.

numerator and denominator go to zero (without actually equating zero). That makes it tricky to define dy and dx as separate entities. (See the margin note, “Leibniz and His Notation.”) Since we really only need to define dy and dx as formal variables, we define them in terms of each other so that their quotient must be the derivative.

DEFINITION Differentials

Let $y = f(x)$ be a differentiable function. The **differential dx** is an independent variable. The **differential dy** is

$$dy = f'(x) dx.$$

Unlike the independent variable dx , the variable dy is always a dependent variable. It depends on both x and dx .

EXAMPLE 5 Finding the Differential dy

Find the differential dy and evaluate dy for the given values of x and dx .

(a) $y = x^5 + 37x$, $x = 1$, $dx = 0.01$ (b) $y = \sin 3x$, $x = \pi$, $dx = -0.02$

(c) $x + y = xy$, $x = 2$, $dx = 0.05$

SOLUTION

(a) $dy = (5x^4 + 37) dx$. When $x = 1$ and $dx = 0.01$, $dy = (5 + 37)(0.01) = 0.42$.

(b) $dy = (3 \cos 3x) dx$. When $x = \pi$ and $dx = -0.02$,
 $dy = (3 \cos 3\pi)(-0.02) = 0.06$.

(c) We could solve explicitly for y before differentiating, but it is easier to use implicit differentiation:

$$\begin{aligned} d(x + y) &= d(xy) \\ dx + dy &= xdy + ydx && \text{Sum and Product Rules in differential form} \\ dy(1 - x) &= (y - 1)dx \\ dy &= \frac{(y - 1)dx}{1 - x} \end{aligned}$$

When $x = 2$ in the original equation, $2 + y = 2y$, so y is also 2. Therefore

$$dy = \frac{(2 - 1)(0.05)}{(1 - 2)} = -0.05.$$

Now Try Exercise 15.

If $dx \neq 0$, then the quotient of the differential dy by the differential dx is equal to the derivative $f'(x)$ because

$$\frac{dy}{dx} = \frac{f'(x)dx}{dx} = f'(x).$$

We sometimes write

$$df = f'(x) dx$$

in place of $dy = f'(x) dx$, calling df the **differential of f** . For instance, if $f(x) = 3x^2 - 6$, then

$$df = d(3x^2 - 6) = 6x dx.$$

Fan Chung Graham (1949–)



“Don’t be intimidated!” is Dr. Fan Chung Graham’s advice to young women considering careers in mathematics. Fan Chung Graham came to the United States

from Taiwan to earn a Ph.D. in Mathematics from the University of Pennsylvania. She worked in the field of combinatorics at Bell Labs and Bellcore, and then, in 1994, returned to her alma mater as a Professor of Mathematics. Her research interests include spectral graph theory, discrete geometry, algorithms, and communication networks.

Every differentiation formula like

$$\frac{d(u + v)}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \text{or} \quad \frac{d(\sin u)}{dx} = \cos u \frac{du}{dx}$$

has a corresponding differential form like

$$d(u + v) = du + dv \quad \text{or} \quad d(\sin u) = \cos u du.$$

EXAMPLE 6 Finding Differentials of Functions

(a) $d(\tan 2x) = \sec^2(2x) d(2x) = 2 \sec^2 2x dx$

(b) $d\left(\frac{x}{x+1}\right) = \frac{(x+1)dx - x d(x+1)}{(x+1)^2} = \frac{xdx + dx - xdx}{(x+1)^2} = \frac{dx}{(x+1)^2}$

Now Try Exercise 23.

Estimating Change with Differentials

Suppose we know the value of a differentiable function $f(x)$ at a point a and we want to predict how much this value will change if we move to a nearby point $a + dx$. If dx is small, f and its linearization L at a will change by nearly the same amount (Figure 5.47). Since the values of L are simple to calculate, calculating the change in L offers a practical way to estimate the change in f .

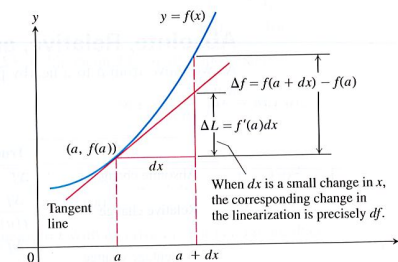


Figure 5.47 Approximating the change in the function f by the change in the linearization of f .

In the notation of Figure 5.47, the change in f is

$$\Delta f = f(a + dx) - f(a).$$

The corresponding change in L is

$$\begin{aligned} \Delta L &= L(a + dx) - L(a) \\ &= \underbrace{f(a) + f'(a)[(a + dx) - a]}_{L(a + dx)} - \underbrace{f(a)}_{L(a)} \\ &= f'(a) dx. \end{aligned}$$

Thus, the differential $df = f'(x) dx$ has a geometric interpretation: The value of df at $x = a$ is ΔL , the change in the linearization of f corresponding to the change dx .

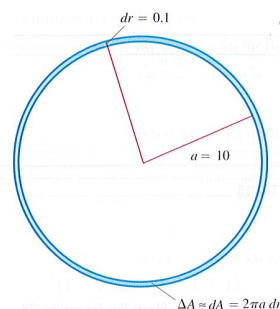


Figure 5.48 When dr is small compared with a , as it is when $dr = 0.1$ and $a = 10$, the differential $dA = 2\pi a dr$ gives a good estimate of ΔA . (Example 7)

Differential Estimate of Change

Let $f(x)$ be differentiable at $x = a$. The approximate change in the value of f when x changes from a to $a + dx$ is

$$df = f'(a) dx.$$

EXAMPLE 7 Estimating Change with Differentials

The radius r of a circle increases from $a = 10$ m to 10.1 m (Figure 5.48). Use dA to estimate the increase in the circle's area A . Compare this estimate with the true change ΔA , and find the approximation error.

SOLUTION

Since $A = \pi r^2$, the estimated increase is

$$dA = A'(a) dr = 2\pi a dr = 2\pi(10)(0.1) = 2\pi \text{ m}^2.$$

The true change is

$$\Delta A = \pi(10.1)^2 - \pi(10)^2 = (102.01 - 100)\pi = 2.01\pi \text{ m}^2.$$

The approximation error is $\Delta A - dA = 2.01\pi - 2\pi = 0.01\pi \text{ m}^2$.

Now Try Exercise 27.

Absolute, Relative, and Percentage Change

As we move from a to a nearby point $a + dx$, we can describe the change in f in three ways:

	True	Estimated
Absolute change	$\Delta f = f(a + dx) - f(a)$	$df = f'(a) dx$
Relative change	$\frac{\Delta f}{f(a)}$	$\frac{df}{f(a)}$
Percentage change	$\frac{\Delta f}{f(a)} \times 100$	$\frac{df}{f(a)} \times 100$

EXAMPLE 8 Changing Tires

Inflating a bicycle tire changes its radius from 12 inches to 13 inches. Use differentials to estimate the absolute change, the relative change, and the percentage change in the perimeter of the tire.

SOLUTION

Perimeter $P = 2\pi r$, so $\Delta P \approx dP = 2\pi dr = 2\pi(1) = 2\pi \approx 6.28$.

The absolute change is approximately 6.3 inches.

The relative change (when $P(12) = 24\pi$) is approximately $2\pi/24\pi \approx 0.08$.

The percentage change is approximately 8 percent.

Now Try Exercise 31.

Another way to interpret the change in $f(x)$ resulting from a change in x is the effect that an error in estimating x has on the estimation of $f(x)$. We illustrate this in Example 9.

Why It's Easy to Estimate Change in Perimeter

Note that the true change in Example 8 is $P(13) - P(12) = 26\pi - 24\pi = 2\pi$, so the differential estimate in this case is perfectly accurate! Why? Since $P = 2\pi r$ is a linear function of r , the linearization of P is the same as P itself. It is useful to keep in mind that local linearity is what makes estimation by differentials work.

EXAMPLE 9 Estimating the Earth's Surface Area

Suppose the earth were a perfect sphere and we determined its radius to be 3959 ± 0.1 miles. What effect would the tolerance of ± 0.1 mi have on our estimate of the earth's surface area?

SOLUTION

The surface area of a sphere of radius r is $S = 4\pi r^2$. The uncertainty in the calculation of S that arises from measuring r with a tolerance of dr miles is

$$dS = 8\pi r dr.$$

With $r = 3959$ and $dr = 0.1$, our estimate of S could be off by as much as

$$dS = 8\pi(3959)(0.1) \approx 9950 \text{ mi}^2,$$

to the nearest square mile, which is about the area of the state of Maryland.

Now Try Exercise 37.

EXAMPLE 10 Determining Tolerance

About how accurately should we measure the radius r of a sphere to calculate the surface area $S = 4\pi r^2$ within 1% of its true value?

SOLUTION

We want any inaccuracy in our measurement to be small enough to make the corresponding increment ΔS in the surface area satisfy the inequality

$$|\Delta S| \leq \frac{1}{100} S = \frac{4\pi r^2}{100}.$$

We replace ΔS in this inequality by its approximation

$$dS = \left(\frac{dS}{dr}\right) dr = 8\pi r dr.$$

This gives

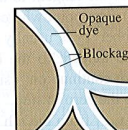
$$|8\pi r dr| \leq \frac{4\pi r^2}{100}, \quad \text{or} \quad |dr| \leq \frac{1}{8\pi r} \cdot \frac{4\pi r^2}{100} = \frac{1}{2} \cdot \frac{r}{100} = 0.005r.$$

We should measure r with an error dr that is no more than 0.5% of the true value.

Now Try Exercise 45.

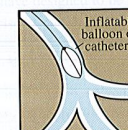
Angiography

An opaque dye is injected into a partially blocked artery to make the inside visible under X-rays. This reveals the location and severity of the blockage.



Angioplasty

A balloon-tipped catheter is inflated inside the artery to widen it at the blockage site.



EXAMPLE 11 Unclogging Arteries

In the late 1830s, the French physiologist Jean Poiseuille ("pwa-ZOY") discovered the formula we use today to predict how much the radius of a partially clogged artery has to be expanded to restore normal flow. His formula,

$$V = kr^4,$$

says that the volume V of fluid flowing through a small pipe or tube in a unit of time at a fixed pressure is a constant times the fourth power of the tube's radius r . How will a 10% increase in r affect V ?

SOLUTION

The differentials of r and V are related by the equation

$$dV = \frac{dV}{dr} dr = 4kr^3 dr.$$

continued

The relative change in V is

$$\frac{dV}{V} = \frac{4kr^3 dr}{kr^4} = 4 \frac{dr}{r}.$$

The relative change in V is 4 times the relative change in r , so a 10% increase in r will produce a 40% increase in the flow. **Now Try Exercise 47.**

Sensitivity to Change

The equation $df = f'(x) dx$ tells how sensitive the output of f is to a change in input at different values of x . The larger the value of f' at x , the greater the effect of a given change dx .

EXAMPLE 12 Finding Depth of a Well

You want to calculate the depth of a well from the equation $s = 16t^2$ by timing how long it takes a heavy stone you drop to splash into the water below. How sensitive will your calculations be to a 0.1-sec error in measuring the time?

SOLUTION

The size of ds in the equation

$$ds = 32t dt$$

depends on how big t is. If $t = 2$ sec, the error caused by $dt = 0.1$ is only

$$ds = 32(2)(0.1) = 6.4 \text{ ft.}$$

Three seconds later at $t = 5$ sec, the error caused by the same dt is

$$ds = 32(5)(0.1) = 16 \text{ ft.}$$

Now Try Exercise 49.

Newton's Method

Newton's method is a numerical technique for approximating a zero of a function with zeros of its linearizations. Under favorable circumstances, the zeros of the linearizations converge rapidly to an accurate approximation. Many calculators use the method because it applies to a wide range of functions and usually gets results in only a few steps. Here is how it works.

To find a solution of an equation $f(x) = 0$, we begin with an initial estimate x_1 , found either by looking at a graph or by simply guessing. Then we use the tangent to the curve $y = f(x)$ at $(x_1, f(x_1))$ to approximate the curve (Figure 5.49). The point where the tangent crosses the x -axis is the next approximation x_2 . The number x_2 is usually a better approximation to the solution than is x_1 . The point where the tangent to the curve at $(x_2, f(x_2))$ crosses the x -axis is the next approximation x_3 . We continue on, using each approximation to generate the next, until we are close enough to the zero to stop.

There is a formula for finding the $(n + 1)$ st approximation x_{n+1} from the n th approximation x_n , which is shown in the following procedure for Newton's method.

Procedure for Newton's Method

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

See Figure 5.50.

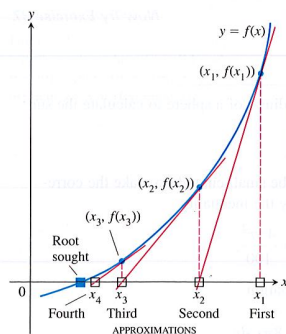


Figure 5.49 Usually the approximations rapidly approach an actual zero of $y = f(x)$.

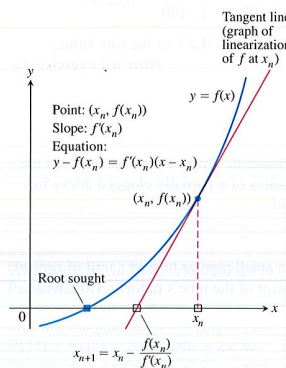


Figure 5.50 From x_n we go up to the curve and follow the tangent line down to find x_{n+1} .

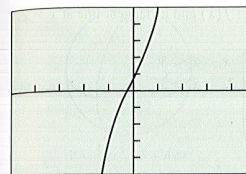


Figure 5.51 A calculator graph of $y = x^3 + 3x + 1$ suggests that -0.3 is a good first guess at the zero to begin Newton's method. (Example 13)

EXAMPLE 13 Using Newton's Method

Use Newton's method to solve $x^3 + 3x + 1 = 0$.

SOLUTION

Let $f(x) = x^3 + 3x + 1$, then $f'(x) = 3x^2 + 3$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + 3x_n + 1}{3x_n^2 + 3}.$$

The graph of f in Figure 5.51 suggests that $x_1 = -0.3$ is a good first approximation to the zero of f in the interval $-1 \leq x \leq 0$. Then,

$$\begin{aligned} x_1 &= -0.3, \\ x_2 &= -0.322324159, \\ x_3 &= -0.3221853603, \\ x_4 &= -0.3221853546. \end{aligned}$$

The x_n for $n \geq 5$ all appear to equal x_4 on the calculator we used for our computations. We conclude that the solution to the equation $x^3 + 3x + 1 = 0$ is about -0.3221853546 . **Now Try Exercise 53.**

Newton's Method May Fail

Newton's method does not work if $f'(x_1) = 0$. In that case, choose a new starting point.

Newton's method does not always converge. For instance (see Figure 5.52), successive approximations $r - h$ and $r + h$ can go back and forth between these two values, and no amount of iteration will bring us any closer to the zero r .

If Newton's method does converge, it converges to a zero of f . However, the method may converge to a zero that is different from the expected one if the starting value is not close enough to the zero sought. Figure 5.53 shows how this might happen.

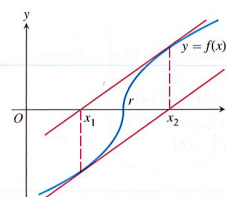


Figure 5.52 The graph of the function

$$f(x) = \begin{cases} -\sqrt{r-x}, & x < r \\ \sqrt{x-r}, & x \geq r. \end{cases}$$

If $x_1 = r - h$, then $x_2 = r + h$.

Successive approximations go back and forth between these two values, and Newton's method fails to converge.

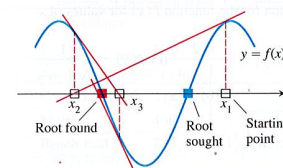


Figure 5.53 Newton's method may miss the zero you want if you start too far away.

Quick Review 5.5 (For help, go to Sections 3.3, 4.1, and 4.4.)

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator*.

In Exercises 1 and 2, find dy/dx .

1. $y = \sin(x^2 + 1)$

2. $y = \frac{x + \cos x}{x + 1}$

In Exercises 3 and 4, solve the equation graphically.

3. $xe^{-x} + 1 = 0$

4. $x^3 + 3x + 1 = 0$

In Exercises 5 and 6, let $f(x) = xe^{-x} + 1$. Write an equation for the line tangent to f at $x = c$.

5. $c = 0$

6. $c = -1$

7. Find where the tangent line in (a) Exercise 5 and (b) Exercise 6 crosses the x -axis.

8. Let $g(x)$ be the function whose graph is the tangent line to the graph of $f(x) = x^3 - 4x + 1$ at $x = 1$. Complete the table.

x	$f(x)$	$g(x)$
0.7		
0.8		
0.9		
1		
1.1		
1.2		
1.3		

Section 5.5 Exercises

- In Exercises 1–6, (a) find the linearization $L(x)$ of $f(x)$ at $x = a$. (b) How accurate is the approximation $L(a + 0.1) \approx f(a + 0.1)$? See the comparisons following Example 1.

- $f(x) = x^3 - 2x + 3$, $a = 2$
- $f(x) = \sqrt{x^2 + 9}$, $a = -4$
- $f(x) = x + \frac{1}{x}$, $a = 1$
- $f(x) = \ln(x + 1)$, $a = 0$
- $f(x) = \tan x$, $a = \pi$
- $f(x) = \cos^{-1} x$, $a = 0$
- Show that the linearization of $f(x) = (1 + x)^k$ at $x = 0$ is $L(x) = 1 + kx$
- Use the linearization $(1 + x)^k \approx 1 + kx$ to approximate the following. State how accurate your approximation is.
 - $(1.002)^{100}$
 - $\sqrt[3]{1.009}$

In Exercises 9 and 10, use the linear approximation $(1 + x)^k \approx 1 + kx$ to find an approximation for the function $f(x)$ for values of x near zero.

- (a) $f(x) = (1 - x)^6$ (b) $f(x) = \frac{2}{1 - x}$ (c) $f(x) = \frac{1}{\sqrt{1 + x}}$
- (a) $f(x) = (4 + 3x)^{1/3}$ (b) $f(x) = \sqrt{2 + x^2}$
(c) $f(x) = \sqrt[3]{1 - \frac{1}{2 + x}}$

In Exercises 11–14, approximate the root by using a linearization centered at an appropriate nearby number.

- $\sqrt[3]{101}$
- $\sqrt[3]{26}$
- $\sqrt[3]{998}$
- $\sqrt{80}$

In Exercises 15–22, (a) find dy , and (b) evaluate dy for the given value of x and dx .

- $y = x^3 - 3x$, $x = 2$, $dx = 0.05$
- $y = \frac{2x}{1 + x^2}$, $x = -2$, $dx = 0.1$
- $y = x^2 \ln x$, $x = 1$, $dx = 0.01$
- $y = x\sqrt{1 - x^2}$, $x = 0$, $dx = -0.2$
- $y = e^{\sin x}$, $x = \pi$, $dx = -0.1$

In Exercises 9 and 10, graph $y = f(x)$ and its tangent line at $x = c$.

- $c = 1.5$, $f(x) = \sin x$
- $c = 4$, $f(x) = \begin{cases} -\sqrt{3 - x}, & x < 3 \\ \sqrt{x - 3}, & x \geq 3 \end{cases}$

20. $y = 3 \csc\left(1 - \frac{x}{3}\right)$, $x = 1$, $dx = 0.1$

21. $y + xy - x = 0$, $x = 0$, $dx = 0.01$

22. $2y = x^2 - xy$, $x = 2$, $dx = -0.05$

In Exercises 23–26, find the differential.

23. $d(\sqrt{1 - x^2})$

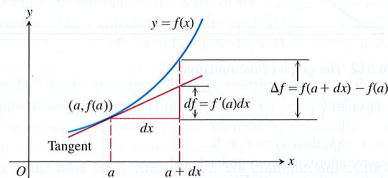
24. $d(e^{5x} + x^5)$

25. $d(\arctan 4x)$

26. $d(8^x + x^8)$

In Exercises 27–30, the function f changes value when x changes from a to $a + dx$. Find

- the true change $\Delta f = f(a + dx) - f(a)$.
- the estimated change $df = f'(a)dx$.
- the approximation error $|\Delta f - df|$.



27. $f(x) = x^2 + 2x$, $a = 0$, $dx = 0.1$

28. $f(x) = x^3 - x$, $a = 1$, $dx = 0.1$

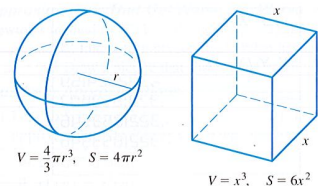
29. $f(x) = x^{-1}$, $a = 0.5$, $dx = 0.05$

30. $f(x) = x^4$, $a = 1$, $dx = 0.01$

In Exercises 31–36, write a differential formula that estimates the given change in volume or surface area. Then use the formula to estimate the change when the independent variable changes from 10 cm to 10.05 cm.

31. **Volume** The change in the volume $V = (4/3)\pi r^3$ of a sphere when the radius changes from a to $a + dr$

32. **Surface Area** The change in the surface area $S = 4\pi r^2$ of a sphere when the radius changes from a to $a + dr$

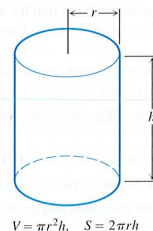


33. **Volume** The change in the volume $V = x^3$ of a cube when the edge lengths change from a to $a + dx$

34. **Surface Area** The change in the surface area $S = 6x^2$ of a cube when the edge lengths change from a to $a + dx$

35. **Volume** The change in the volume $V = \pi r^2 h$ of a right circular cylinder when the radius changes from a to $a + dr$ and the height does not change

36. **Surface Area** The change in the lateral surface area $S = 2\pi rh$ of a right circular cylinder when the height changes from a to $a + dh$ and the radius does not change



In Exercises 37–40, use differentials to estimate the maximum error in measurement resulting from the tolerance of error in the independent variable. Express answers to the nearest tenth, since that is the precision used to express the tolerance.

- The area of a circle with radius 10 ± 0.1 in.
- The volume of a sphere with radius 8 ± 0.3 in.
- The volume of a cube with side 15 ± 0.2 cm.
- The area of an equilateral triangle with side 20 ± 0.5 cm.
- Linear Approximation** Let f be a function with $f(0) = 1$ and $f'(x) = \cos(x^2)$.
 - Find the linearization of f at $x = 0$.
 - Estimate the value of f at $x = 0.1$.
 - Writing to Learn** Do you think the actual value of f at $x = 0.1$ is greater than or less than the estimate in part (b)? Explain.

42. **Expanding Circle** The radius of a circle is increased from 2.00 to 2.02 m.

- Estimate the resulting change in area.
- Estimate as a percentage of the circle's original area.

43. **Growing Tree** The diameter of a tree was 10 in. During the following year, the circumference increased 2 in. About how much did the tree's diameter increase? the tree's cross-section area?

44. **Percentage Error** The edge of a cube is measured as 10 cm with an error of 1%. The cube's volume is to be calculated from this measurement. Estimate the percentage error in the volume calculation.

45. **Tolerance** About how accurately should you measure the side of a square to be sure of calculating the area to within 2% of its true value?

46. **Tolerance** (a) About how accurately must the interior diameter of a 10-m-high cylindrical storage tank be measured to calculate the tank's volume to within 1% of its true value?

- (b) About how accurately must the tank's exterior diameter be measured to calculate the amount of paint it will take to paint the side of the tank to within 5% of the true amount?

47. **Minting Coins** A manufacturer contracts to mint coins for the federal government. The coins must weigh within 0.1% of their ideal weight, so the volume must be within 0.1% of the ideal volume. Assuming the thickness of the coins does not change, what is the percentage change in the volume of the coin that would result from a 0.1% increase in the radius?

48. **Tolerance** The height and radius of a right circular cylinder are equal, so the cylinder's volume is $V = \pi h^3$. The volume is to be calculated with an error of no more than 1% of the true value. Find approximately the greatest error that can be tolerated in the measurement of h , expressed as a percentage of h .

49. **Estimating Volume** You can estimate the volume of a sphere by measuring its circumference with a tape measure, dividing by 2π to get the radius, then using the radius in the volume formula. Find how sensitive your volume estimate is to a 1/8-in. error in the circumference measurement by filling in the table below for spheres of the given sizes. Use differentials when filling in the last column.

Sphere Type	True Radius	Tape Error	Radius Error	Volume Error
Orange	2 in.	1/8 in.		
Melon	4 in.	1/8 in.		
Beach Ball	7 in.	1/8 in.		

50. **Estimating Surface Area** Change the heading in the last column of the table in Exercise 49 to "Surface Area Error" and find how sensitive the measure of surface area is to a 1/8-in. error in estimating the circumference of the sphere.

51. **The Effect of Flight Maneuvers on the Heart** The amount of work done by the heart's main pumping chamber, the left ventricle, is given by the equation

$$W = PV + \frac{V\delta v^2}{2g},$$

where W is the work per unit time, P is the average blood pressure, V is the volume of blood pumped out during the unit of time, δ ("delta") is the density of the blood, v is the average velocity of the exiting blood, and g is the acceleration of gravity.

When P , V , δ , and v remain constant, W becomes a function of g , and the equation takes the simplified form

$$W = a + \frac{b}{g} \quad (a, b \text{ constant}).$$

As a member of NASA's medical team, you want to know how sensitive W is to apparent changes in g caused by flight maneuvers, and this depends on the initial value of g . As part of your investigation, you decide to compare the effect on W of a given change dg on the moon, where $g = 5.2 \text{ ft/sec}^2$, with the effect the same change dg would have on Earth, where $g = 32 \text{ ft/sec}^2$. Use the simplified equation above to find the ratio of dW_{moon} to dW_{Earth} .



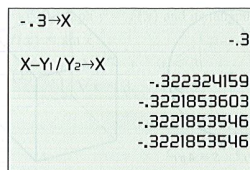
- 52. Measuring Acceleration of Gravity** When the length L of a clock pendulum is held constant by controlling its temperature, the pendulum's period T depends on the acceleration of gravity g . The period will therefore vary slightly as the clock is moved from place to place on the earth's surface, depending on the change in g . By keeping track of ΔT , we can estimate the variation in g from the equation $T = 2\pi(L/g)^{1/2}$ that relates T , g , and L .

- (a) With L held constant and g as the independent variable, calculate dT and use it to answer parts (b) and (c).
 (b) **Writing to Learn** If g increases, will T increase or decrease? Will a pendulum clock speed up or slow down? Explain.
 (c) A clock with a 100-cm pendulum is moved from a location where $g = 980 \text{ cm/sec}^2$ to a new location. This increases the period by $dT = 0.001 \text{ sec}$. Find dg and estimate the value of g at the new location.

Using Newton's Method on Your Calculator

A nice way to get your calculator to perform the calculations in Newton's method. Try it with the function $f(x) = x^3 + 3x + 1$ from Example 5.

- Enter the function in Y1 and its derivative in Y2.
- On the home screen, store the initial guess into x . For example, using the initial guess in Example 5, you would type $-.3 \rightarrow X$.
- Type $X - Y1/Y2 \rightarrow X$ and press the ENTER key over and over. Watch as the numbers converge to the zero of f . When the values stop changing, it means that your calculator has found the zero to the extent of its displayed digits, as shown in the following figure.



- Experiment with different initial guesses and repeat Steps 2 and 3.
- Experiment with different functions and repeat Steps 1 through 3. Compare each final value you find with the value given by your calculator's built-in zero-finding feature.

In Exercises 53–56, use Newton's method to estimate all real solutions of the equation. Make your answers accurate to 6 decimal places.

53. $x^3 + x - 1 = 0$ 54. $x^4 + x - 3 = 0$
 55. $x^2 - 2x + 1 = \sin x$ 56. $x^4 - 2 = 0$

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

- 57. True or False** Newton's method will not find the zero of $f(x) = x/(x^2 + 1)$ if the first guess is greater than 1. Justify your answer.
- 58. True or False** If u and v are differentiable functions, then $d(uv) = du \, dv$. Justify your answer.
- 59. Multiple Choice** What is the linearization of $f(x) = e^x$ at $x = 1$?
 (A) $y = e$ (B) $y = ex$ (C) $y = e^x$
 (D) $y = x - e$ (E) $y = e(x - 1)$
- 60. Multiple Choice** If $y = \tan x$, $x = \pi$, and $dx = 0.5$, what does dy equal?
 (A) -0.25 (B) -0.5 (C) 0 (D) 0.5 (E) 0.25
- 61. Multiple Choice** If Newton's method is used to find the zero of $f(x) = x - x^3 + 2$, what is the third estimate if the first estimate is 1?
 (A) $-\frac{3}{4}$ (B) $\frac{3}{2}$ (C) $\frac{8}{5}$ (D) $\frac{18}{11}$ (E) 3
- 62. Multiple Choice** If the linearization of $y = \sqrt[3]{x}$ at $x = 64$ is used to approximate $\sqrt[3]{66}$, what is the percentage error?
 (A) 0.01% (B) 0.04% (C) 0.4% (D) 1% (E) 4%

Explorations

- 63. Newton's Method** Suppose your first guess in using Newton's method is lucky in the sense that x_1 is a root of $f(x) = 0$. What happens to x_2 and later approximations?
- 64. Oscillation** Show that if $h > 0$, applying Newton's method to
- $$f(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ \sqrt{-x}, & x < 0 \end{cases}$$
- leads to $x_2 = -h$ if $x_1 = h$, and to $x_2 = h$ if $x_1 = -h$. Draw a picture that shows what is going on.

- 65. Approximations That Get Worse and Worse** Apply Newton's method to $f(x) = x^{1/3}$ with $x_1 = 1$, and calculate x_2 , x_3 , x_4 , and x_5 . Find a formula for $|x_n|$. What happens to $|x_n|$ as $n \rightarrow \infty$? Draw a picture that shows what is going on.

66. Quadratic Approximations

- (a) Let $Q(x) = b_0 + b_1(x - a) + b_2(x - a)^2$ be a quadratic approximation to $f(x)$ at $x = a$ with the properties:
- $Q(a) = f(a)$,
 - $Q'(a) = f'(a)$,
 - $Q''(a) = f''(a)$.

Determine the coefficients b_0 , b_1 , and b_2 .

- (b) Find the quadratic approximation to $f(x) = 1/(1 - x)$ at $x = 0$.
 (c) Graph $f(x) = 1/(1 - x)$ and its quadratic approximation at $x = 0$. Then ZOOM IN on the two graphs at the point $(0, 1)$. Comment on what you see.
 (d) Find the quadratic approximation to $g(x) = 1/x$ at $x = 1$. Graph g and its quadratic approximation together. Comment on what you see.
 (e) Find the quadratic approximation to $h(x) = \sqrt{1 + x}$ at $x = 0$. Graph h and its quadratic approximation together. Comment on what you see.
 (f) What are the linearizations of f , g , and h at the respective points in parts (b), (d), and (e)?

- 67. Multiples of Pi** Store any number as X in your calculator. Then enter the command $X - \tan(X) \rightarrow X$ and press the ENTER key repeatedly until the displayed value stops changing. The result is always an integral multiple of π . Why is this so? [Hint: These are zeros of the sine function.]

Extending the Ideas

- 68. Formulas for Differentials** Verify the following formulas.

- (a) $d(c) = 0$ (c a constant)
 (b) $d(cu) = c \, du$ (c a constant)
 (c) $d(u + v) = du + dv$
 (d) $d(u \cdot v) = u \, dv + v \, du$
 (e) $d\left(\frac{u}{v}\right) = \frac{v \, du - u \, dv}{v^2}$
 (f) $d(u^n) = nu^{n-1} \, du$

- 69. Linearization** Show that the approximation of $\tan x$ by its linearization at the origin must improve as $x \rightarrow 0$ by showing that

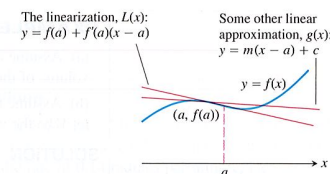
$$\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1.$$

70. The Linearization Is the Best Linear Approximation

Suppose that $y = f(x)$ is differentiable at $x = a$ and that $g(x) = m(x - a) + c$ (m and c constants). If the error $E(x) = f(x) - g(x)$ were small enough near $x = a$, we might think of using g as a linear approximation of f instead of the linearization $L(x) = f(a) + f'(a)(x - a)$. Show that if we impose on g the conditions

- $E(a) = 0$,
- $\lim_{x \rightarrow a} \frac{E(x)}{x - a} = 0$,

then $g(x) = f(a) + f'(a)(x - a)$. Thus, the linearization gives the only linear approximation whose error is both zero at $x = a$ and negligible in comparison with $(x - a)$.



- 71. Writing to Learn** Find the linearization of $f(x) = \sqrt{x} + 1 + \sin x$ at $x = 0$. How is it related to the individual linearizations for $\sqrt{x} + 1$ and $\sin x$?

- 72. Formula for Newton's Method** Derive the formula for finding the $(n + 1)$ st approximation x_{n+1} from the n th approximation x_n . [Hint: Write the point-slope equation for the tangent line to the curve at $(x_n, f(x_n))$. See Figure 5.50 on page 244. Then set $y = 0$ and solve for x .]